数分B3习题课讲义

> 3.5 $A \subset \mathbb{R}$, A' 至多可数，则 A 至多可数

$A \setminus A'$ 无极限点，否则该点属于 A' ，矛盾。
在 $C2$ 空间（有可数基），孤立点集至多可数，故 $A \setminus A'$ 至多可数，进而 A 至多可数。

> 3.6 $[0,1]$ cannot be the disjoint union of countably many closed sets.

Lemma 1 : $F \subset \mathbb{R}$ is a closed set. C is a component of F. Then $\partial F \cap C \neq \phi$

Let $x_0 \in C$. Suppose that C is disjoint from ∂F. Then there is an open-closed set $A \subset Int(F) \subset F$ and $x_0 \in A$. Then by the definition of subspace topology, $\exists U \subset \mathbb{R}$, which is open, such that $A = U \cap F$. Since $A \cap Int(F) = \phi$, we have $A = U \cap Int(F)$. Thus A is open in $\mathbb{R}$. But A is also closed in F, and F closed in $\mathbb{R}$. So A is closed in $\mathbb{R}$, which implies $A = \mathbb{R}$. And it is impossible.

Lemma 2 : Closed interval $F = \bigsqcup F_i$, F_i's are non-empty. Then for all $i \neq j$, there is a closed interval $C \subset \mathbb{R}$ satisfying
$C \cap F_i = \phi, C \cap F_j \neq \phi$, and $C \cap F_k, k = 1, 2, \cdots$ has at least two non-empty sets.

If F_i is empty then we take $C = \mathbb{R}$. Thus we can assume that F_i's are not empty. Take $j \neq i$. By the property of (T4), there are disjoint open set U, V such that $F_i \subset U$ and $F_j \subset V$. Let $x \in F_j$ and C the component of x in the subspace $\overline{V} \subset U^C$. C is a closed interval and $C \cap F_i = \phi, C \cap F_j \neq \phi$. By lemma 1, $C \cap \partial \overline{V} \neq \phi$. Because $F_j \subset V$, there is another k such that F_k intersects with $C \cap \partial \overline{V}$, thus it intersects with C.

Let a closed interval $F = \bigsqcup F_i$, and each F_i is closed. For each i, we can find a closed interval $C_i \subset C_{i-1}$ which doesn't intersect with $\bigsqcup_{k=1}^{i} (F_i \cap C_{i-1})$. So we get a decreasing closed set sequence $C_1 \supset C_2 \supset \cdots$, thus $\cap_{k=1}^{\infty} C_k \neq \phi$. While $\cap_{k=1}^{\infty} C_k \cap F = \phi$ by the construction, which is a contradiction.

Remarks :

1. 注意区分闭集和闭区间
2. 开集可以写成可数个开区间的并，但闭区间不行

> 3.9 $\mathbb{Q}$ is not a G_δ set, i.e. $\mathbb{Q}$ cannot be the intersection of countably many open sets.

Suppose $\mathbb{Q} = \cap_{i=1}^{\infty} U_i = \{q_1, q_2, \cdots\}$, where U_i's are open sets. Take a closed interval $[a_1 - \epsilon_1, a_1 + \epsilon_1] \subset U_1 \setminus \{q_1\}$. Similarly we can find a closed interval $[a_k - \epsilon_k, a_k + \epsilon_k] \subset (U_k \setminus \{q_k\}) \cap (a_{k-1} - \epsilon_{k-1}, a_{k-1} + \epsilon_{k-1})$. By the compactness, $V = \cap_{k=1}^{\infty} [a_k - \epsilon_k, a_k + \epsilon_k] \neq \phi$. Because $V \subset U_i$ for each i, $U \subset \mathbb{Q}$. But $q_k \notin V$ by the construction.

Remarks:

1. Let $U_m = \cup_{n=1}^{\infty} (q_n - 1/2^{n+m}, q_n + 1/2^{n+m})$. Is $\cap_{m=1}^{\infty} U_m$ equal to $\mathbb{Q}$? What can we infer from this example?
2. There are "a lot of" points in U_n^C !

> Example : A topological space is called a **Baire space** if every intersection of a countable collection of open dense sets is also dense.

Fact :

1. Any complete metric space is a Baire space.
2. Any compact Hausdorff space is a Baire space
3. Any locally compact Hausdorff space is a Baire space

> Example : Cantor set

$$C = [0,1] \setminus \cup_{n=1}^{\infty} \cup_{k=0}^{3^{n-1}-1} (\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$$

Fact :

1. C is closed.
2. Evrey point in the Cantor set is a limit point, i.e. $C = C'$.
3. $C = \{\sum_{k=1}^{\infty} \frac{2a_k}{3^k} : a_k \in \{0,1\}\}$
4. We can define a map

$$g : \{0,1\}^{\mathbb{N}} \to [0,1], (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{2a_k}{3^k}$$

   It is not difficult to show that g is a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and C.
5. (Cantor Function) By the binary representation, we have a homeomorphism

$$h : \{0,1\}^{\mathbb{N}} \to [0,1], (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

   Let $f = h \circ g^{-1} : C \to \mathbb{R}$. Since C^C is the union of disjoint countably many open intervals (a_i, b_i), and $f(a_i) = f(b_i)$, so we can connect $(a_i, f(a_i))$ and $(b_i, f(b_i))$ with a horizontal line segment for each i. Thus we get a monotonic function $f_C : [0,1] \to [0,1]$, which is called Cantor Function.

> 4.3 In (X, ρ), continuity is equivalent to sequence continuity.

"$\Rightarrow$" : $f : X \to Y$ is continuous. For any $x_n \to x_0$, and any neighborhood V of $f(x_0)$, $f^{-1}(V)$ is a neighborhood of x_0. Therefore $\exists N, \forall n > N, x_n \in f^{-1}(V) \Rightarrow f(x_n) \in V$. So $f(x_n) \to f(x_0)$.

"$\Leftarrow$" : It suffices to prove that for any $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. **By the property of metric space**, for each $a \in \overline{A}, \exists a_n \in A$ such that $a_n \to a$. Therefore $f(a_n) \to f(a)$, and $f(a) \in \overline{f(A)}$. So $f(\overline{A}) \subset \overline{f(A)}$.

Remarks :

1. Without "metric", $x \in \overline{A}$ maynot be the limit of a sequence of $x_n \in A$. For example, in $(\mathbb{R}, \mathcal{T}_{cocountable})$, let A be any uncountable set. Then $\overline{A} = \mathbb{R}$ since the intersection of A

and every open set in $\mathbb{R}$ has uncountably many elements. But $x_n \to x$ if and only if $\exists N, \forall m, n > N, a_m = a_n$.

2. $Id : (\mathbb{R}, \mathcal{T}_{cocountable}) \to (\mathbb{R}, \mathcal{T}_{discrete})$ is sequence continuous but not continuous.

> Example : upper semi-continuous topology
> We can define a topology on $\mathbb{R}$
>
> $$\mathcal{T}_{u.s.c} = \{(-\infty, a) : a \in \mathbb{R}\}$$
>
> We say a function $f : (X, \mathcal{T}) \to \mathbb{R}$ is **upper semi-continuous** if $\forall x_0 \in X, \forall \epsilon > 0, \exists U \in \mathcal{N}(x_0)$ such that $\forall x \in U, f(x) < f(x_0) + \epsilon$.

Fact : The following statements are equivalent:

1. f is upper semi-continuous
2. $f : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_{u.s.c})$ is continuous.
3. If $(X, \mathcal{T}) = (\mathbb{R}, \mathcal{T}_{usual})$, $\overline{lim}_{x \to x_0} f(x) \leq f(x_0)$

Similarly, we can define **lower semi continuous** and **lower semi-continuous topology**.